Optimal vs the Stiffened Circular Plate

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The simply supported circular plate has been optimized by other investigators for both uniform pressure and uniform compression. The objective of the optimization in both cases is to improve the load carrying capability of the plate by finding the proper distribution of material. In the present paper the author presents a comparison between the "optimal plate" and a stiffened plate for both load cases. Closed form solutions are given for the stiffened plate by using the following special geometry: a) the stiffeners are one-sided and with constant eccentricity and b) the smeared extensional and flexual stiffnesses are the same along both the radial and circumferential directions and constant. The comparison shows that this particular stiffened plate geometry has better load carrying capability than the corresponding optimal plate geometry of the same volume and radius.

Nomenclature

A_r, A_θ	= stiffener cross-sectional area
D	= flexural stiffness of the plate $[Eh^3/12(1-\mu^2)]$
D_r^S, D_θ^S	= flexural stiffness of the stiffeners
E	= Young's Modulus for the plate
E_r, E_θ	= Young's Modulus for the stiffeners
E^P	= plate extensional stiffness
E_r^S, E_θ^S	= stiffener extensional stiffness
e_r, e_θ	= stiffener eccentricity
G	= shear modulus for the plate
h	= plate thickness
$I_{r_{CG}}, I_{\theta_{CG}}$	= second moment of stiffner cross-sectional area
l_r, l_θ	= stiffener spacing
$M_{rr}, M_{\theta\theta}, M_{r\theta}$	= moment resultants
$N_{rr}, N_{\theta\theta}, N_{r\theta}$	= stress resultants
$\bar{N}$	= applied stress resultant (uniform compression)
q_0	= applied uniform pressure
r, θ	= polar coordinates
R	= radius of circular plate
r_s	= stiffness ratio $(\bar{w}_U/\bar{w}_{ST})$
u_r, u_θ, w	= reference surface displacement components
V	= volume of plate material
$\bar{v}$	= volume per radian of plate material
$\bar{w}$	= average displacement $\left(=\frac{2}{R^2}\int_0^R wr\,dr\right)$
α	$= E/12(1-\mu^2)$
$\varepsilon_{rr}, \varepsilon_{\theta\theta}, \gamma_{r\theta}$	= reference surface strains
$\kappa_{rr}, \kappa_{\theta\theta}, \kappa_{r\theta}$	= reference surface changes in curvature and torsion
λ	= ratio of extensional stiffnesses, stiffener to plate
μ	= Poisson's ratio
ρ_0	= ratio of flexural stiffnesses, stiffener to plate
ρ^2	$= \bar{N}/D[1+\rho_0+12(\lambda/1+\lambda)(e/h)^2]$
ψ	= stress function
OPT	= optimal geometry
ST	= stiffened geometry
U	= uniform thickness geometry

1. Introduction

THE constant demand for light weight efficient structural systems, especially in the aerospace industry, has lead many investigators to the field of structural optimization. A large number of these investigators including the author have taken the micro approach and dealt with the optimization of basic structural elements such as columns, beams, and plates. A survey paper by Sheu and Prager[1] presents a wealth of references dealing with beams and columns, and a limited number dealing with plates. A recent paper by the present author[2] and his collaborators shows that the optimization of the column has been successfully accomplished. Similarly, the beam problem has received wide attention but with lesser success.

In any structural optimization problem one must clearly specify a) the design objective and b) the geometric and behavioral constraints. In structural elements which are subject to instability, e.g., the design objective could be the maximization of the buckling load. For this same problem the geometric constraints are specified volume and length or physical dimensions, and the behavioral constraint is that the stress should not exceed a specified value.

In accordance with this objective and constraints, Frauenthal[3] reported solutions to the following problem. Given a thin circular flat plate of specified volume and radius, loaded by uniform radial thrust, distribute the material (thickness) in such a way as to maximize the buckling load, subject to the constraint that the maximum allowable prebuckling stress remains smaller than a prescribed value. Among the important results of this investigation is that the buckling load for a simply supported solid cross-sectional geometry, under the assumption of axisymmetric behavior and without any constraint on the prebuckling stress, is increased by 49% when compared to a uniform thickness plate of the same volume and radius.

The simply supported circular plate has also been optimized for a uniform transverse load. Huang[4] reports the solution to the following particular problem. Given a thin circular plate of specified volume and radius, simply supported around the circumference and subjected to a uniform transverse loading of given magnitude, distribute the material (thickness) in such a way as to minimize the average deflection. Another way of stating and accomplishing the same objective is to maximize the load per unit deflection. With this objective in mind and without any behavioral constraints he shows that the load carrying capability of the optimal circular plate is increased by 58% as compared to a uniform thickness plate of the same volume and radius.

In both problems mentioned above the investigators resorted to calculus of variations, in order to derive the field equations. In this type of an approach (for details see Refs. 3 and 4) a functional is introduced in accordance with the design objective and the volume constraint (isoperimetric problem), and necessary conditions (equilibrium, optimality condition, etc.) are arrived at, which govern the behavior of the design objective. Because of the mathematical tool employed the family of functions, from which the solution is sought (admissible comparison functions), has imposed limitations. In Ref. 3, it is shown that the admissible comparison functions are of class C^1, i.e., single-

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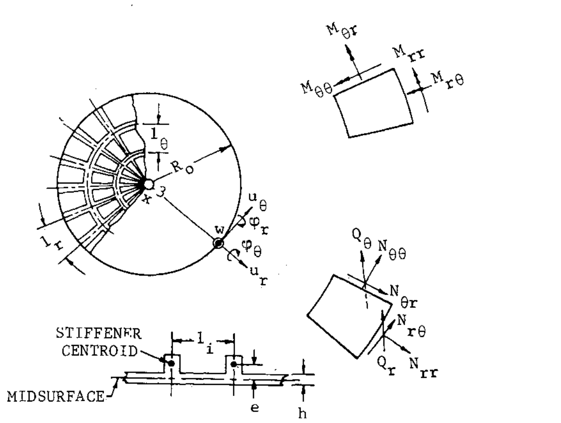

Fig. 1 The stiffened circular plate geometry and sign convection.

valued, continuous with continuous first derivatives. Similarly, in Ref. 4, it is shown that the admissible comparison functions are also of class C^1. Now the question arises "Is it possible that the true solution lies outside functions of class C^1?" In other words, if one does not limit the thickness distribution to be governed by single-valuedness and continuity could one obtain increases in the buckling load and in the transverse load higher than those reported in Refs. 3 and 4?

This question motivated the author to compare critical loads and transverse loads of a stiffened thin circular plate simply supported around its circumference to those of a unfirom thickness plate of the same volume and radius. The latter used the same basis of comparison as in Ref. 4.

It should be pointed out at this point that a number of authors[5-7] have demonstrated that stiffened configurations have better load carrying capability (for a number of load cases) when compared to uniform thickness configuration. Some of these authors have proceeded to optimize the stiffened geometry by using a number of search techniques (mathematical programing).[6,7]

2. The Stiffened Circular Plate

The circular plate is simply supported along its circumference and is stiffened in both the circumferential and radial directions such that a) the stiffeners are both one-sided, b) the stiffener eccentricity is the same for all stiffeners and constant (note that when the ecccentricity is zero the same amount of stiffening is present on both sides of the plate for all stiffeners), and c) the smeared extensional and flexural stiffnesses are the same along both the radial and circumferential directions and constant. The geometry and sign convention are given in Fig. 1.

First, the equilibrium equations are derived for the stiffened circular plate loaded by both transverse loading q_0 and radial thrust around the circumference. Then, when each load case is considered separately the equations are properly modified.

In deriving the equations, the plate midsurface is used as a reference surface and the von Kârmân kinematic relations are employed. In addition, the needed relations between the stress and moments resultants and the reference surface strains and changes in curvature and torsion are obtained by employing the "smearing" technique (see Baruch and Singer[8]). These relations are

$$
\begin{aligned}
N_{rr} &(E^P+E^S)\varepsilon_{rr}+\mu E^P\varepsilon_{\theta\theta}+eE^S\kappa_{rr}\\
N_{\theta\theta} &= \mu E^P\varepsilon_{rr}+(E^P+E^S)\varepsilon_{\theta\theta}+eE^S\kappa_{\theta\theta}\\
N_{r\theta} &= Gh\gamma_{r\theta}\\
M_{rr} &= (D+D^S+e^2E^S)\kappa_{rr}+\mu D\kappa_{\theta\theta}+eE^S\varepsilon_{rr}\\
M_{\theta\theta} &= \mu D\kappa_{rr}+(D+D^S+e^2E^S)\kappa_{\theta\theta}+eE^S\varepsilon_{\theta\theta}\\
M_{r\theta} &= D(1-\mu)\kappa_{r\theta}
\end{aligned}
\tag{1}
$$

where

$$
\begin{aligned}
&E^P = Eh/(1-\mu^2);\quad E_r^S = E_r A_r/l_r;\quad E_\theta^S = E_\theta A_\theta/l_\theta\\
&E_r^S = E_\theta^S = E^S;\quad D = Eh^3/12(1-\mu^2);\quad D_r^S = E_r I_{rCG}/l_r\\
&D_\theta^S = E_\theta I_{\theta CG}/l_\theta;\quad D_r^S = D_\theta^S = D^S;\quad e_e = e_\theta = e
\end{aligned}
\tag{2}
$$

and E, E_r, and E_θ denote the Young's Moduli for the plate, radial, and circumferential stiffeners, respectively; A_i and $I_{i_{CG}}$ the stiffener areas and second moment of the stiffener areas (CG implies centroidal axes); l_i the stiffener spacings; ε_{ij} and κ_{ij} the reference surface strains and changes in curvature and torsion; μ the Poisson's ratio for the plate material; and e_i the stiffener eccentricities.

The equilibrium equations in terms of the stress resultants N_{ij}, M_{ij}, and transverse displacement components w are given below

$$
\begin{aligned}
&N_{rr}+rN_{rr,r}+N_{r\theta,\theta}-N_{\theta\theta}=0\\
&N_{\theta\theta}+rN_{r\theta,r}+2N_{r\theta}=0\\
&rM_{rr,rr}+2M_{rr,r}+2M_{r\theta,r\theta}-M_{\theta\theta,r}+(1/r)(M_{\theta\theta,\theta}+2M_{r\theta})_{,\theta}+\\
&\qquad N_{\theta\theta}w_{,r}+rN_{rr}w_{,rr}+2N_{r\theta}w_{,r\theta}-\\
&\qquad (2/r)N_r w_{,\theta}+(1/r)N_{\theta\theta}w_{,\theta\theta}=-q_0 r
\end{aligned}
\tag{3}
$$

where the comma denotes partial differentiation with respect to the indices.

Uniform Pressure

Since the lateral loading is a uniform load of q_0 psi and the geometry is axisymmetric it is safe to assume that the response is also axisymmetric. For pure bending theory the linearized compatibility equation is given by

$$(r\varepsilon_{\theta\theta})_{,rr}-\varepsilon_{rr,r}=0 \tag{4}$$

Next, by introducing a stress function ψ such that the in-plane equilibrium equations are identically satisfied, the compatibility and third equilibrium equations form the field equations and are given by

compatibility

$$\nabla^4\psi = [E^P e\lambda\mu/(1+\lambda)]\nabla^4 w \tag{5}$$

equilibrium

$$\left[1+\rho_0+e^2\lambda\frac{E^P}{D}\left(\frac{1+\mu}{1+\mu+\lambda}\right)\right]\nabla^4 w-\frac{e\lambda}{D(1+\mu+\lambda)}\nabla^4 w = q_0/D \tag{6}$$

where

$$\lambda = E^S/E^P \quad\text{and}\quad \rho_0 = D_S/D \tag{7}$$

The associated boundary conditions for a simply supported plate are

$$\left.\begin{aligned}&w=0;\quad M_{rr}=0\\ &\text{and either } N_{rr}=0 \quad\text{or}\quad u_r=0\end{aligned}\right\}\text{at } r=R \tag{8}$$

In the solution for this problem the weaker of the in-plane boundary conditions is used since it leads to larger displacements, and this will support further the conclusion that the stiffened plate subjected to a uniform load q_0 has a smaller average deflection than the corresponding optimal plate of the same volume and radius and subjected to the same loading q_0. The boundary conditions, Eqs. (8), in terms of w and ψ are

$$w(R)=0 \tag{9a}$$

$$
\begin{aligned}
&\left[1+\rho_0+e^2\lambda\frac{E^P}{D}-\frac{e^2\lambda^2E^P(1+\lambda)}{D\{(1+\lambda)^2-\mu^2\}}\right]w_{,rr}(R)+\\
&\mu\left[1+\frac{e^2\lambda^2E^P}{\{D(1+\lambda)^2-\mu^2\}}\right]w_{,r}(R)/R+\\
&\frac{e\lambda\mu}{D[(1+\lambda)^2-\mu^2]}\psi_{,rr}(R)=0
\end{aligned}
\tag{9b}
$$

$$\psi_{,r}(R)=0$$

The solution is accomplished by first eliminating ψ from the equilibrium equation (6) through the use of the compatibility equation (5). Then solving this equation for w one obtains a solution in terms of two arbitrary constants (the other constants are set equal to zero in order to satisfy finiteness at $4 = 0$). Finally, using this expression for w in the compatibility equation one solves for ψ and a new additional constant is introduced. These three constants are evaluated by using the boundary conditions, Eqs. (9). The final expressions for w and ψ are given by

$$w = \frac{q_0}{64\bar{D}}\left\{r^4 - 2r^2R^2\left[3(1+\rho_0)+\mu+ 12\lambda\left(\frac{e}{h}\right)^2\frac{3(1+\mu+\lambda)+\lambda\mu}{(1+\lambda)(1+\lambda+\mu)}\right]\Big/\left[1+\rho_0+\mu+ 12\left(\frac{e}{h}\right)^2\frac{\lambda(1+\mu)}{1+\lambda+\mu}\right] + R^4\left[5(1+\rho_0)+\mu+ 12\lambda\left(\frac{e}{h}\right)^2\frac{5(1+\lambda+\mu)+\lambda\mu}{(1+\lambda)(1+\lambda+\mu)}\right]\Big/\left[1+\rho_0+\mu+ 12\lambda\left(\frac{e}{h}\right)^2\frac{1+\mu}{1+\lambda+\mu}\right]\right\} \tag{10}$$

$$\psi = \frac{q_0 E^P e\lambda\mu}{64\bar{D}(1+\lambda)}r^2(r^2-2R^2) \tag{11}$$

where

$$\bar{D} = D\left[1+\rho_0+12\left(\frac{e}{h}\right)^2\frac{\lambda}{1+\lambda}\right] \tag{12}$$

It is easily seen from Eq. (10) that if one sets $\rho_0 = e = \lambda = 0$ he obtains the well-known expression for the isotropic case

$$w = \frac{q_0(R^2-r^2)}{64D}\left[\frac{5+\mu}{1+\mu}R^2 - r^2\right] \tag{13}$$

Uniform Compression

The solution to this problem has been reported by the present author and Blackmon.[9] For the weaker of the two in-plane boundary conditions the characteristic equation is given by Eq. (14) of Ref. 9. This equation is

$$(\rho R)J_0(\rho R) - \left[1-\mu\left\{12\left(\frac{e}{h}\right)^2\frac{\lambda}{\lambda+1}\left\{\frac{\lambda}{1+\lambda+\mu}\right\}+1\right\}\Big/\left\{1+\rho_0+ 12\left(\frac{e}{h}\right)^2\frac{\lambda}{\lambda+1}\right\}\right]J_1(\rho R) = 0 \tag{14}$$

where J_0 and J_1 are Bessel functions of the first kind of orders zero and one, respectively, and

$$\rho^2 = (\bar{N}/D)\left[1+\rho_0+12\left(\frac{e}{h}\right)^2\frac{\lambda}{1+\lambda}\right] \tag{15}$$

3. Comparisons with Uniform Geometries

In this section a comparison is given between the response of the stiffened plate and that of a uniform thickness plate of the same volume and radius, and material.

The volume of a uniform thickness plate is given by

$$\bar{V} = \int_0^{2\pi}\int_0^R h_U\, r\,dr\,d\theta = \pi R^2 h_U \tag{16}$$

through the introduction of a new volume parameter $\bar{v}$ (volume per radian) one obtains

$$\bar{v} = \bar{V}/2\pi = R^2 h_U/2 \tag{17}$$

It is now possible to express the bending stiffness of the uniform thickness plate D_U in the following manner

$$D_U = Eh_U{}^3/12(1-\mu^2) = \alpha\, 8\bar{v}^3/R^6 \tag{18}$$

where $\alpha = E/12(1-\mu^2)$.

Similarly, the volume of the stiffened plate is equal to the sum of the volume of the plate of thickness h and the volume of the stiffeners, or

$$\bar{V} = \int_0^{2\pi}\int_0^R hr\,dr\,d\theta + \int_0^{2\pi}\int_0^R\left(\frac{A_\theta}{l_\theta}+\frac{A_r}{l_r}\right)r\,dr\,d\theta \tag{19}$$

If one assumes that the stiffener and the plate materials are the same then

$$A_\theta/l_\theta = A_r/l_r = \lambda h/(1-\mu^2) \tag{20}$$

and

$$\bar{v} = \bar{V}/2\pi = (R^2h/2)[1+2\lambda/(1-\mu^2)] \tag{21}$$

The bending stiffness of the plate in the stiffened geometry is given by

$$D = \alpha\, 8\bar{v}^3/R^6(1+2\lambda/1-\mu^2)^3 \tag{22}$$

Uniform Pressure

For comparison purposes a parameter r_s is introduced (same as in Ref. 4) which denotes the ratio of the stiffness of the stiffened plate to that of a uniform plate of the same volume, radius, and material.

$$r_s = \bar{w}_U/\bar{w}_{ST} \tag{23}$$

where $\bar{w}_U$ and $\bar{w}_{ST}$ denote the average deflection of the uniform thickness and stiffened plates, respectively.

Since

$$\bar{w} = \frac{2}{R^2}\int_0^R wr\,dr \tag{24}$$

use of Eqs. (10) and (13) in Eq. (23) yields

$$\bar{w}_{ST} = (q_0 R^4/64\bar{D})\left[\tfrac{1}{3} - A + B\right] \tag{25}$$

where

$$A = \left\{3(1+\rho_0)+\mu+12\lambda\left(\frac{e}{h}\right)^2\frac{3(1+\lambda+\mu)+\lambda\mu}{(1+\lambda)(1+\lambda+\mu)}\right\}\Big/\left\{1+\rho_0+ \mu+12\lambda\left(\frac{e}{h}\right)^2\frac{1+\mu}{1+\lambda+\mu}\right\} \tag{26a}$$

and

$$B = \left[5(1+\rho_0)+\mu+12\lambda\left(\frac{e}{h}\right)^2\frac{5(1+\lambda+\mu)+\lambda\mu}{(1+\lambda)(1+\lambda+\mu)}\right]\Big/\left[1+\rho_0+ \mu+12\lambda\left(\frac{e}{h}\right)^2\frac{1+\mu}{1+\lambda+\mu}\right] \tag{26b}$$

and

$$\bar{w}_U = 1.8712\, q_0 R^4/64D_U \tag{27}$$

Finally by employing Eqs. (12) (for $\bar{D}$), (18) (for D_U), and (23) (for D) one may write ($\mu = 0.3$)

$$\frac{1}{r_r} = \frac{\bar{w}_{ST}}{\bar{w}_U} = \frac{(1+2\lambda/1-\mu^2)^3(1/3-A+B)}{1.8712[1+\rho_0+12(e/h)^2(\lambda/1+\lambda)]} \tag{28}$$

The stiffness ratio is computed for three geometries and plotted vs e/h in Fig. 2. The three geometries are 1) light stiffening

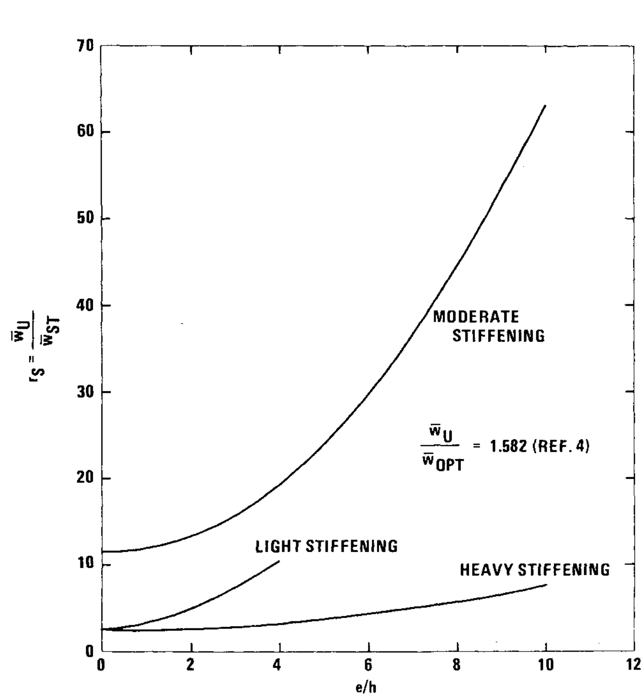

Fig. 2 Stiffness ratio vs stiffener eccentricity.

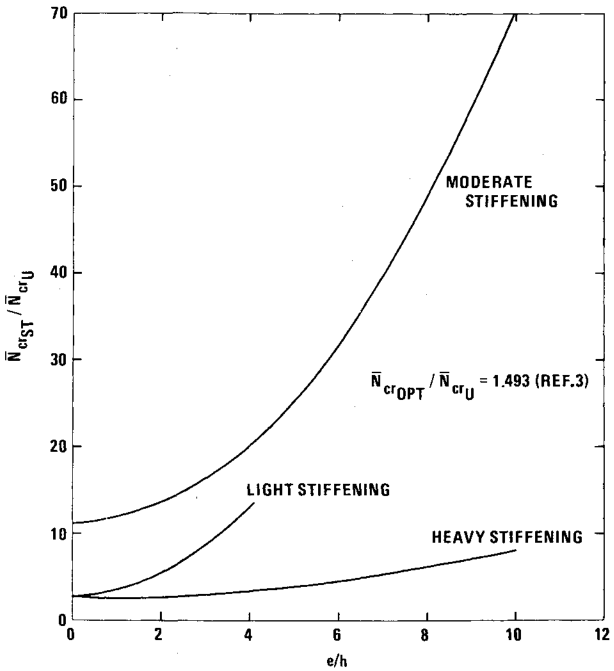

Fig. 3 Ratio of critical loads (stiffened to uniform thickness geometries) vs stiffener eccentricity.

$\lambda = 0.1$ and $\rho_0 = 5$, 2) moderate stiffening $\lambda = 0.25$ and $\rho_0 = 50$, and 3) heavy stiffening $\lambda = 2$ and $\rho_p = 500$. Light stiffening implies that most of the extensional stiffness is provided by the plate, moderate stiffening implies that the extensional stiffness is provided by both the plate and the stiffeners almost equally, and heavy stiffening implies that most of the extensional stiffness is provided by the stiffeners.

Uniform Compression

For this load case the ratio of $\bar{N}_{cr_{ST}}$ to $\bar{N}_{cr_U}$ is computed for the same three geometries and plotted vs e/h in Fig. 3. The critical load for a simply supported uniform thickness circular plate is given by the well-known expression

$$\bar{N}_{cr_U} = 4.2 D_U / R^2 \tag{29}$$

The expression for the simply supported stiffened circular plate is given by

$$\bar{N}_{cr_{ST}} = \frac{(\rho R)_{cr}{}^2 D}{R^2}\left[1+\rho_0+12\left(\frac{e}{h}\right)^2 \frac{\lambda}{1+\lambda}\right] \tag{30}$$

As before, through the use of Eqs. (18) and (22), the ratio of the critical loads is

$$\frac{\bar{N}_{cr_{ST}}}{\bar{N}_{cr_U}} = \frac{(\rho R)_{cr}{}^2}{4.2}\frac{[1+\rho_0+12(e/h)^2(\lambda/1+\lambda)]}{[1+2\lambda/1-\mu^2]^3} \tag{31}$$

where $(\rho R)_{cr}$ is the solution of Eq. (14) for each geometry.

4. Conclusions

It is apparent from the graphically presented results (see Figs. 2 and 3) of this investigation that the stiffened plate chosen has better response characteristics than the "optimal" plate. At this point it should be noted that special care is needed in designing these stiffened plates to make certain that there are no local instabilities. For example, the skin between adjacent stiffeners should not be allowed to wrinkle because this reduces the extensional and flexural stiffness contributions of the plate. In addition, the spacings and the geometry of the stiffeners should be such that the stiffeners do not buckle locally.

Another important conclusion drawn from Figs. 2 and 3 is that for a given radius and volume, if one uses the stiffened configuration used in this paper, it is better to distribute the material in such a way that the resulting configuration is one of moderate stiffening rather than either light or heavy stiffening. This observation indicates there is an optimum stiffened configuration.

Finally, it is concluded that if true optimization is to be accomplished one should consider the stiffened plate and make extensive parametric studies in order to evaluate the effect of certain parameters such as stiffener sizes and spacings on the overall response of the plate.

This can be best accomplished through the use of good search techniques in connection with this constrained optimization problem (like those in Refs. 6 and 7).

References

[1] Sheu, C. Y. and Prager, W., "Recent Developments in Optimal Structural Design," *Applied Mechanics Reviews*, Vol. 21, No. 10, Oct. 1968, pp. 985–992.

[2] Simitses, G. J., Kamat, M. P., and Smith, C. V., Jr., "The Strongest Column by the Finite Element Displacement Method," AIAA Paper 72-141, San Diego, Calif., 1972.

[3] Frauenthal, J. C., "Constrained Optimal Design of Circular Plates Against Buckling," Rept. SM-50, April 1971, Div. of Engineering and Applied Physics, Harvard Univ., Cambridge, Mass.

[4] Huang, N. C., "Optimal Design of Elastic Structures for Maximum Stiffness," *International Journal of Solids and Structures*, Vol. 4, 1968, pp. 689–700.

[5] Singer, J. and Barruch, M., "Recent Studies on Optimization for Elastic Stability of Cylindrical and Conical Shells," *Aerospace Proceedings 1966, Proceedings of the 5th Congress of the International Council of the Aeronautical Sciences and Royal Aeronautical Centenary Congress*, McMillan, London, Sept. 1966, pp. 751–758.

[6] Schmit, L. A., Morrow, W. M., and Kicher, T. P., "A Structural Synthesis Capability for Integrally Stiffened Cylindrical Shells," AIAA Paper 68-327, Palm Springs, Calif., 1968.

[7] Jones, R. T. and Hague, D. S., "Application of Multivariable Search Techniques to Structural Design Optimization," CR-2038, June 1972, NASA.

[8] Baruch, M. and Singer, J., "Effect of Eccentricity of Stiffeners on the General Instability of Stiffened Cylindrical Shells Under Hydrostatic Pressure," *Journal of the Mechanical Engineering Sciences*, Vol. 5, No. 1, 1963, pp. 23–27.

[9] Simitses, G. J. and Blackmon, C. M., "Buckling of Eccentrically Stiffened Thin Circular Plates," *AIAA Journal*, Vol. 7, No. 6, June 1969, pp. 1200–1202.